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A study on the fourth q -Painlevé equation

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Abstract

A q -difference analogue of the fourth Painlevé equation is proposed. Its symmetry structure and some particular solutions are investigated.

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1. Introduction

The importance of six Painlevé equations and their solutions in mathematics and mathematical physics is widely accepted. The discrete analogue of Painlevé equations was first recognized in [6–8], where they appear as Schlesinger transformations of the Painlevé equations. After the discovery of the discrete analogue of the Painlevé property which is now called the singularity confinement property [4], many second-order difference equations, including q -difference equations, have been identified as the discrete Painlevé equations [5, 16]. By investigating the Lax pairs, particular solutions including τ functions and bilinear equations, and so on [3], it has been gradually recognized that discrete Painlevé equations admit similar properties to original Painlevé equations.

The purpose of this paper is to introduce a q -difference analogue of the fourth Painlevé equation P_{IV} . This equation, called the q -Painlevé IV equation qP_{IV} below, shares many characteristic properties with the original P_{IV} . In particular, it admits the action of the affine Weyl group of type $A_2^{(1)}$ as a group of Bäcklund transformations. Furthermore, it has an analogue of classical solutions expressible by the continuous q -Hermite–Weber functions and rational solutions corresponding to those of P_{IV} studied in [10, 13, 15].

The plan of this paper is as follows. We introduce our qP_{IV} in section 2 and describe its symmetry structure in terms of the affine Weyl group $W(A_2^{(1)})$. In section 3, we construct a q -analogue of classical solutions along each reflection hyperplane from a seed solution which is described by a discrete Riccati equation. Each particular solution of this class is expressed in terms of a Toeplitz type determinant of continuous q -Hermite–Weber functions. In section 4, we discuss some related topics including the relationship between our qP_{IV} and Sakai's Mul.6 system [17] and the ultra-discrete limit of qP_{IV} .

2. Symmetric form of a q -Painlevé equation qP_{IV}

2.1. q -Painlevé equation qP_{IV}

In this paper, we consider the following discrete system including three dependent variables f_0 , f_1 and f_2 and three parameters a_0 , a_1 and a_2 :

$$\begin{aligned}\bar{a}_0 &= a_0 & \bar{a}_1 &= a_1 & \bar{a}_2 &= a_2 \\ \bar{f}_0 &= a_0 a_1 f_1 \frac{1 + a_2 f_2 + a_2 a_0 f_2 f_0}{1 + a_0 f_0 + a_0 a_1 f_0 f_1} \\ \bar{f}_1 &= a_1 a_2 f_2 \frac{1 + a_0 f_0 + a_0 a_1 f_0 f_1}{1 + a_1 f_1 + a_1 a_2 f_1 f_2} \\ \bar{f}_2 &= a_2 a_0 f_0 \frac{1 + a_1 f_1 + a_1 a_2 f_1 f_2}{1 + a_2 f_2 + a_2 a_0 f_2 f_0}\end{aligned}\quad (1)$$

where $\bar{}$ stands for the discrete time evolution. We shall also use the notation $t(x) = \bar{x}$ when we regard $x \rightarrow \bar{x}$ as a transformation of variables. The inverse transformation of t is given by

$$\begin{aligned}\underline{a}_0 &= a_0 & \underline{a}_1 &= a_1 & \underline{a}_2 &= a_2 \\ \underline{f}_0 &= \frac{f_2}{a_0 a_1} \frac{a_0 a_1 + a_0 f_1 + f_0 f_1}{a_2 a_0 + a_2 f_0 + f_2 f_0} \\ \underline{f}_1 &= \frac{f_0}{a_1 a_2} \frac{a_1 a_2 + a_1 f_2 + f_1 f_2}{a_0 a_1 + a_0 f_1 + f_0 f_1} \\ \underline{f}_2 &= \frac{f_1}{a_2 a_0} \frac{a_2 a_0 + a_2 f_0 + f_2 f_0}{a_1 a_2 + a_1 f_2 + f_1 f_2}.\end{aligned}\quad (2)$$

We introduce a constant q by setting $a_0 a_1 a_2 = q$. Noting that the product $f_0 f_1 f_2$ can be regarded as the independent variable, we introduce a variable c such that $f_0 f_1 f_2 = qc^2$ and $\bar{c} = qc$. If we regard f_j ($j = 0, 1, 2$) as functions in c , equation (1) thus represents a system of q -difference equations for the unknown functions $f_j = f_j(c)$ ($j = 0, 1, 2$) with parameters a_j ($j = 0, 1, 2$) such that $a_0 a_1 a_2 = q$. As we shall see in section 1.3 below, this q -difference system has a continuous limit to the *symmetric form* of the fourth Painlevé equation P_{IV} . For this reason, we refer to equation (1) as the symmetric form of the fourth q -Painlevé equation qP_{IV} .

2.2. Bäcklund transformations

The discrete system equation (1) admits the action of the (extended) affine Weyl group $\bar{W} = \langle s_0, s_1, s_2, \pi \rangle$ of type $A_2^{(1)}$ as a group of Bäcklund transformations. In what follows we denote by \bar{W} the group generated by the generators s_0, s_1, s_2, π and the fundamental relations

$$s_i^2 = 1 \quad (s_i s_{i+1})^3 = 1 \quad \pi^3 = 1 \quad \pi s_i = s_{i+1} \pi \quad (i = 0, 1, 2) \quad (3)$$

where the indices are understood as elements of $\mathbb{Z}/3\mathbb{Z}$.

We define the action of s_0, s_1, s_2 and π on the parameters a_0, a_1 and a_2 by the following formulae:

$$\begin{aligned}s_0(a_0) &= a_0^{-1} & s_0(a_1) &= a_1 a_0 & s_0(a_2) &= a_2 a_0 \\ s_1(a_0) &= a_0 a_1 & s_1(a_1) &= a_1^{-1} & s_1(a_2) &= a_2 a_1 \\ s_2(a_0) &= a_0 a_2 & s_2(a_1) &= a_1 a_2 & s_2(a_2) &= a_2^{-1} \\ \pi(a_0) &= a_1 & \pi(a_1) &= a_2 & \pi(a_2) &= a_0.\end{aligned}\quad (4)$$

Note that this definition is a multiplicative version of the standard realization of \tilde{W} on the simple roots α_j defined through $a_j = q^{\alpha_j}$ ($j = 0, 1, 2$). We now define the action of s_0, s_1, s_2 and π on the f -variables as follows:

$$\begin{aligned} s_0(f_0) &= f_0 & s_0(f_1) &= f_1 \frac{a_0 + f_0}{1 + a_0 f_0} & s_0(f_2) &= f_2 \frac{1 + a_0 f_0}{a_0 + f_0} \\ s_1(f_0) &= f_0 \frac{1 + a_1 f_1}{a_1 + f_1} & s_1(f_1) &= f_1 & s_1(f_2) &= f_2 \frac{a_1 + f_1}{1 + a_1 f_1} \\ s_2(f_0) &= f_0 \frac{a_2 + f_2}{1 + a_2 f_2} & s_2(f_1) &= f_1 \frac{1 + a_2 f_2}{a_2 + f_2} & s_2(f_2) &= f_2 \\ \pi(f_0) &= f_1 & \pi(f_1) &= f_2 & \pi(f_2) &= f_0. \end{aligned} \tag{5}$$

We remark that the action of s_i on the a -variables and the f -variables is given by

$$s_i(a_j) = a_j a_i^{-a_{ij}} \quad s_i(f_j) = f_j \left(\frac{a_i + f_i}{1 + a_i f_i} \right)^{u_{ij}} \quad (i, j = 0, 1, 2) \tag{6}$$

respectively, where $A = (a_{ij})_{i,j=0}^2$ is the generalized Cartan matrix of type $A_2^{(1)}$ and $U = (u_{ij})_{i,j=0}^2$ is an orientation matrix of the corresponding Dynkin diagram:

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad U = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}. \tag{7}$$

The following theorem can be verified by direct computation.

Theorem 2.1. *The transformations s_0, s_1, s_2 and π of the a -variables and the f -variables, defined by (4) and (5), generate the extended affine Weyl group $\tilde{W} = \langle s_0, s_1, s_2, \pi \rangle$ of type $A_2^{(1)}$. Furthermore, they commute with the time evolution t of the fourth q -Painlevé equation qP_{IV} .*

By using the action of the extended affine Weyl group \tilde{W} , we can define the Schlesinger transformations T_1, T_2 and T_3 for qP_{IV} as

$$T_1 = \pi s_2 s_1 \quad T_2 = s_1 \pi s_2 \quad T_3 = s_2 s_1 \pi. \tag{8}$$

Note that $T_i T_j = T_j T_i$ ($i, j = 1, 2, 3$) and $T_1 T_2 T_3 = 1$. The action of T_1 on the variables a_j and f_j is given explicitly as follows:

$$\begin{aligned} T_1(a_0) &= q a_0 & T_1(a_1) &= q^{-1} a_1 & T_1(a_2) &= a_2 \\ T_1(f_0) &= f_1 \frac{(a_0 + f_0)(a_0 + f_0 + a_0 a_2 f_2 + a_0^2 a_2 f_0 f_2)}{(1 + a_0 f_0)(a_0^2 a_2 + a_0 a_2 f_0 + f_2 + a_0 f_0 f_2)} \\ T_1(f_1) &= f_2 \frac{1 + a_0 f_0}{a_0 + f_0} \\ T_1(f_2) &= f_0 \frac{a_0^2 a_2 + a_0 a_2 f_0 + f_2 + a_0 f_0 f_2}{a_0 + f_0 + a_0 a_2 f_2 + a_0^2 a_2 f_0 f_2}. \end{aligned} \tag{9}$$

The corresponding formulae for T_2 and T_3 are obtained by the rotation of indices, since $T_2 = \pi T_1 \pi^{-1}$ and $T_3 = \pi T_2 \pi^{-1}$. Each of these Schlesinger transformations commutes with the time evolution of qP_{IV} and can be regarded as a version of the third q -Painlevé equation. For example, equation (9) is rewritten equivalently as

$$\begin{aligned} T_1(f_1) &= f_2 \frac{1 + a_0 f_0}{a_0 + f_0} = \frac{q c^2}{f_0 f_1} \frac{1 + a_0 f_0}{a_0 + f_0} \\ T_1^{-1}(f_0) &= f_2 \frac{a_1 + f_1}{1 + a_1 f_1} = \frac{q c^2}{f_0 f_1} \frac{a_1 + f_1}{1 + a_1 f_1}. \end{aligned} \tag{10}$$

This is an analogue of the fact that some discrete Painlevé equations arise from Bäcklund transformations of the Painlevé equations [3, 6–8, 12].

2.3. Limit transition to P_{IV}

By introducing a small parameter ε such that $q = e^{-\varepsilon^2/2}$, we set

$$\alpha_i = e^{-\varepsilon^2 \alpha_i / 2} \quad f_i = -e^{-\varepsilon \varphi_i} \quad (j = 0, 1, 2). \quad (11)$$

Then we have

$$\bar{f}_i - f_i = \varepsilon^2 (\varphi_i (\varphi_{i+1} - \varphi_{i+2}) + \alpha_i) + O(\varepsilon^3) \quad (12)$$

for $i = 0, 1, 2$. Passing to the variables α_j and φ_j , we define the derivation $'$ by

$$x' = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\bar{x} - x) \quad (13)$$

for a function x in α_j and φ_j . Then we have $\bar{f}_i - f_i = \varepsilon^2 \varphi_i' + O(\varepsilon^3)$. Hence we obtain

$$\begin{aligned} \alpha_0' &= 0 & \alpha_1' &= 0 & \alpha_2' &= 0 \\ \varphi_0' &= \varphi_0(\varphi_1 - \varphi_2) + \alpha_0 \\ \varphi_1' &= \varphi_1(\varphi_2 - \varphi_0) + \alpha_1 \\ \varphi_2' &= \varphi_2(\varphi_3 - \varphi_1) + \alpha_2. \end{aligned} \quad (14)$$

This differential system is the symmetric form the fourth Painlevé equation P_{IV} [1, 14, 19]. In fact, under the normalization $\varphi_0 + \varphi_1 + \varphi_2 = t$, $\alpha_0 + \alpha_1 + \alpha_2 = 1$, the system (14) is equivalent to the second-order differential equation

$$y'' = \frac{1}{2y} (y')^2 + \frac{3}{2} y^3 - 2ty^2 + \left(\frac{t^2}{2} + \alpha_1 - \alpha_2 \right) y - \frac{\alpha_0^2}{2y} \quad (15)$$

for $y = \varphi_0$, where $' = d/dt$. Through the limiting procedure, the Bäcklund transformations s_0, s_1, s_2 and π for our qP_{IV} also pass to those for the symmetric form of P_{IV} (14) such that

$$\begin{aligned} s_i(\alpha_j) &= \alpha_j - \alpha_i a_{ij} & s_i(\varphi_j) &= \varphi_j + \frac{\alpha_i}{\varphi_i} u_{ij} \\ \pi(\alpha_j) &= \alpha_{j+1} & \pi(\varphi_j) &= \varphi_{j+1} \end{aligned} \quad (i, j = 0, 1, 2) \quad (16)$$

as in [13, 14].

3. Particular solutions for qP_{IV}

In this section, we investigate the q -analogue of classical solutions of P_{IV} and determinant formulae for them. Recall that P_{IV} has two classes of classical solutions [10, 14, 15], those of hypergeometric type, expressed in terms of Hermite–Weber functions, and the rational solutions.

3.1. Continuous q -Hermite–Weber functions as seed solutions

In order to show the parameter dependence explicitly, we rewrite qP_{IV} (1) by operating $T_1^\nu T_2^N$ on (1) ($\nu, N \in \mathbb{Z}$). We denote $T_1^\nu T_2^N(f_i) = f_i(c; \nu, N)$, and we abbreviate the unnecessary arguments depending on the context. Notice that

$$\begin{aligned} T_1^\nu T_2^N(a_0) &= a_0 q^\nu & T_1^\nu T_2^N(a_1) &= a_1 q^{-\nu+N} \\ T_1^\nu T_2^N(a_2) &= a_2 q^{-N} & T_1^\nu T_2^N(c) &= c \end{aligned} \quad (17)$$

and

$$t(a_i) = a_i \quad (i = 0, 1, 2) \quad t(c) = qc \tag{18}$$

where

$$q = a_0 a_1 a_2 \quad c = \left(\frac{f_0 f_1 f_2}{q} \right)^{1/2}. \tag{19}$$

The q -Painlevé IV equation qP_{IV} with respect to the variable c is expressed as

$$f_0(qc) = a_0 a_1 q^N f_1(c) \frac{1 + a_2 q^{-N} f_2(c) + a_2 a_0 q^{v-N} f_2(c) f_0(c)}{1 + a_0 q^v f_0(c) + a_0 a_1 q^N f_0(c) f_1(c)} \tag{20}$$

$$f_1(qc) = a_1 a_2 q^{-v} f_2(c) \frac{1 + a_0 q^v f_0(c) + a_0 a_1 q^N f_0(c) f_1(c)}{1 + a_1 q^{-v+N} f_1(c) + a_1 a_2 q^{-v} f_1(c) f_2(c)} \tag{21}$$

$$f_2(qc) = a_2 a_0 q^{v-N} f_0(c) \frac{1 + a_1 q^{-v+N} f_1(c) + a_1 a_2 q^{-v} f_1(c) f_2(c)}{1 + a_2 q^{-N} f_2(c) + a_2 a_0 q^{v-N} f_2(c) f_0(c)} \tag{22}$$

where the arguments v and N for f_i are suppressed so that $f_i(c) = f_i(c; v, N)$. In what follows, we use similar abbreviations

$$\begin{aligned} f_i(q^k c) &= f_i(q^k c; v, N) & f_i(v+k) &= f_i(c; v+k, N) \\ f_i(N+k) &= f_i(c; v, N+k) \end{aligned} \tag{23}$$

for $k \in \mathbb{Z}$.

First let us consider the case $N = 0$. It is possible to specialize the variables as

$$f_2 = -1 \quad a_2 = 1 \tag{24}$$

consistently. In fact, equation (22) becomes trivial by this specialization, and equations (20) and (21) are reduced to a discrete Riccati type equation

$$f_1(qc) = -\frac{q}{a_0 q^v} \frac{(1 - q^2 c^2) f_1(c) - a_0 q^v c^2}{f_1(c)} \tag{25}$$

and $f_0(c) = -\frac{qc^2}{f_1(c)}$. By putting $f_1(c) = \frac{F_v(c)}{G_v(c)}$ in equation (25), we can solve equation (25) as

$$f_1(c) = -\frac{q}{a_0 q^v} \frac{G_v(qc)}{G_v(c)} \tag{26}$$

with $G_v(c)$ satisfying the linear q -difference equation

$$G_v(q^2 c) = (1 - q^2 c^2) G_v(qc) + a_0^2 q^{2v} c^2 G_v(c). \tag{27}$$

Let us derive contiguity relations to be satisfied by $G_v(c)$ in the direction of v . For this purpose, we consider the Schlesinger transformation T_1 (9),

$$f_0(v+1) = f_1(v) \frac{a_0 q^v + f_0(v)}{1 + a_0 q^v f_0(v)} \frac{a_0 q^v + f_0(v) + a_0 a_2 q^{v-N} f_2(v) + a_0^2 a_2 q^{2v-N} f_0(v) f_2(v)}{a_0^2 a_2 q^{2v-N} + a_0 a_2 q^{v-N} f_0(v) + f_2(v) + a_0 q^v f_0(v) f_2(v)} \tag{28}$$

$$f_1(v+1) = f_2(v) \frac{1 + a_0 q^v f_0(v)}{a_0 q^v + f_0(v)} \tag{29}$$

$$f_2(v+1) = f_0(v) \frac{a_0^2 a_2 q^{2v-N} + a_0 a_2 q^{v-N} f_0(v) + f_2(v) + a_0 q^v f_0(v) f_2(v)}{a_0 q^v + f_0(v) + a_0 a_2 q^{v-N} f_2(v) + a_0^2 a_2 q^{2v-N} f_0(v) f_2(v)}. \tag{30}$$

Applying the specialization (24) and putting $N = 0$, equations (28)–(30) are reduced to a discrete Riccati type equation,

$$f_1(v+1) = -\frac{f_1(v) - a_0 q^{v+1} c^2}{a_0 q^v f_1(v) - qc^2}. \tag{31}$$

Substituting equation (26) into (31), we obtain contiguity relations for $G_\nu(c)$,

$$G_{\nu+1}(qc) = G_\nu(qc) + a_0^2 q^{2\nu} c^2 G_\nu(c) \quad (32)$$

$$G_{\nu+1}(c) = G_\nu(qc) + c^2 G_\nu(c). \quad (33)$$

In particular, we obtain a three-term relation in the direction of ν ,

$$G_{\nu+2}(c) = (c^2 + 1)G_{\nu+1}(c) - c^2(1 - a_0^2 q^{2\nu})G_\nu(c). \quad (34)$$

We note that equations (27) and (34) are derived from equations (32) and (33).

From the above discussion, we have:

Proposition 3.1. qP_{IV} (20)–(22) admits a one-parameter family of particular solutions given by

$$f_0 = c^2 a_0 q^\nu \frac{G_\nu(c)}{G_\nu(qc)} \quad f_1 = -\frac{q}{a_0 q^\nu} \frac{G_\nu(qc)}{G_\nu(c)} \quad f_2 = -1 \quad (35)$$

for $N = 0$, where $G_\nu(c)$ is a function satisfying the contiguity relations (32) and (33).

It is interesting to note that equation (27) admits polynomial solutions in c if $a_0 = q$,

$$G_0(c) = 1$$

$$G_1(c) = c^2 + 1$$

$$G_2(c) = c^4 + (1 + q^2)c^2 + 1$$

$$G_3(c) = c^6 + (1 + q^2 + q^4)c^4 + (1 + q^2 + q^4)c^2 + 1$$

...

It is not difficult to check that

$$G_\nu(c) = \sum_{k=0}^{\nu} \frac{(q^2; q^2)_\nu}{(q^2; q^2)_k (q^2; q^2)_{\nu-k}} c^{2\nu-2k} = \sum_{k=0}^{\nu} \begin{bmatrix} \nu \\ k \end{bmatrix}_{q^2} c^{2\nu-2k} \quad (\nu = 0, 1, 2, \dots) \quad (36)$$

satisfy equations (32) and (33), where $\begin{bmatrix} \nu \\ k \end{bmatrix}_q$ is the q -binomial coefficient. The generating function for these polynomials is given by

$$\frac{1}{(c\lambda; q^2)(\lambda/c; q^2)} = \sum_{\nu=0}^{\infty} \frac{G_\nu(c)}{(q^2; q^2)_\nu} \left(\frac{\lambda}{c}\right)^\nu. \quad (37)$$

The polynomials $H_\nu(x) = c^{-\nu} G_\nu(c)$, $x = \frac{c+c^{-1}}{2}$, are called *the continuous q -Hermite polynomials* [2, 11]. In this sense, $c^{-\nu} G_\nu(c)$ for $a_0 \neq q^l$ ($l \in \mathbb{Z}$) may be regarded as a q -difference analogue of the Hermite–Weber functions, to which hereafter we shall refer as ‘continuous q -Hermite–Weber functions’.

We note that it is possible to obtain ‘higher-order’ solutions for $N \in \mathbb{Z}$ by successive applications of T_2 on the solutions obtained above³. These solutions are expressed rationally in the continuous q -Hermite–Weber functions.

From the rotational symmetry π of qP_{IV} , it is possible to apply other specializations, $(a_0, f_0) = (1, -1)$, $\nu = 0$, or $(a_1, f_1) = (1, -1)$, $\nu - N = 0$, on qP_{IV} (20)–(22) and perform the same procedure as discussed above. Therefore, we obtain:

Theorem 3.2. When $a_i = q^k$ for some $i = 0, 1, 2$ and $k \in \mathbb{Z}$, the fourth q -Painlevé equation $qP_{IV}(1)$ admits a one-parameter family of particular solutions which are expressed rationally by the continuous q -Hermite–Weber functions.

Explicit description of these solutions will be given in the next section.

Note that the hypersurfaces $a_i = q^k$ ($i = 0, 1, 2; k \in \mathbb{Z}$) in the parameter space correspond to the reflection hyperplanes of the affine Weyl group $W = W(A_2^{(1)})$. This is an analogue of a well known result by Okamoto for the classical solutions of the continuous P_{IV} [15].

³ The situation is a little delicate for $N < 0$, but we shall discuss this in section 2.2.3.

3.2. Toeplitz determinants

3.2.1. Determinant formula and bilinear equations. In this section, we prove the following determinant formula for the particular solutions obtained in the previous section.

Theorem 3.3. Let $G_\nu(c)$ be a solution of equations (32) and (33). For each $N \in \mathbb{Z}_{\geq 0}$, we define an $N \times N$ Toeplitz determinant

$$\phi_N^\nu(c) = \det (G_{\nu-i+j}(c))_{i,j=1,\dots,N} \quad (N \in \mathbb{Z}_{>0}) \quad \phi_0^\nu(c) = 1. \quad (38)$$

Then,

$$f_0(c; \nu, N) = a_0 q^\nu c^2 \frac{\phi_{N+1}^\nu(c) \phi_N^\nu(qc)}{\phi_{N+1}^\nu(qc) \phi_N^\nu(c)} \quad (39)$$

$$f_1(c; \nu, N) = -\frac{1}{a_0 q^{\nu+N-1}} \frac{\phi_{N+1}^\nu(qc) \phi_N^{\nu-1}(c)}{\phi_{N+1}^\nu(c) \phi_N^{\nu-1}(qc)} \quad (40)$$

$$f_2(c; \nu, N) = -q^N \frac{\phi_N^{\nu-1}(qc) \phi_N^\nu(c)}{\phi_N^\nu(qc) \phi_N^{\nu-1}(c)} \quad (41)$$

satisfy qP_{IV} (20)–(22) with $a_2 = 1$.

We note that the case of $N = 0$ agrees with proposition 3.1.

Theorem 3.3 is a direct consequence of the following proposition:

Proposition 3.4. $\phi_N^\nu(c)$ satisfies the following bilinear difference equations:

$$a_0^2 c^2 q^{2\nu} \phi_N^\nu(qc) \phi_{N+1}^\nu(c) + \phi_N^\nu(c) \phi_{N+1}^\nu(qc) = q^{-2N} \phi_{N+1}^{\nu+1}(qc) \phi_N^{\nu-1}(c) \quad (42)$$

$$a_0^2 q^{2\nu} \phi_N^\nu(qc) \phi_{N+1}^{\nu+1}(c) - \phi_N^\nu(c) \phi_{N+1}^{\nu+1}(qc) = (a_0^2 q^{2\nu} - q^{2N}) \phi_{N+1}^\nu(qc) \phi_N^{\nu+1}(c) \quad (43)$$

$$\phi_{N+1}^{\nu+1}(qc) \phi_N^\nu(c) - q^{2N} \phi_{N+1}^{\nu+1}(c) \phi_N^\nu(qc) = c^2 (a_0^2 q^{2\nu} - q^{2N}) \phi_{N+1}^\nu(c) \phi_N^{\nu+1}(qc) \quad (44)$$

$$\phi_N^{\nu+1}(qc) \phi_N^\nu(c) - \phi_N^{\nu+1}(c) \phi_N^\nu(qc) = -\phi_{N-1}^{\nu+1}(qc) \phi_{N+1}^{\nu+1}(c) \quad (45)$$

$$\phi_N^{\nu+1}(qc) \phi_N^\nu(c) - q^{2N} \phi_{N+1}^{\nu+1}(c) \phi_N^\nu(qc) = \frac{1}{a_0^2 q^{2\nu} c^2} \phi_{N+1}^{\nu+1}(qc) \phi_{N-1}^\nu(c) \quad (46)$$

$$q^{2N} \phi_N^\nu(c) \phi_{N+1}^\nu(c) - \phi_{N+1}^{\nu+1}(c) \phi_N^{\nu-1}(c) = -a_0^2 c^2 q^{2(\nu-1)} \phi_{N+1}^\nu(q^{-1}c) \phi_N^\nu(qc) \quad (47)$$

$$\phi_N^\nu(c) \phi_{N+1}^{\nu+1}(c) + c^2 (a_0^2 q^{2\nu} - q^{2N}) \phi_{N+1}^\nu(c) \phi_N^{\nu+1}(c) = q^{-2N} \phi_N^\nu(q^{-1}c) \phi_{N+1}^{\nu+1}(qc) \quad (48)$$

$$a_0^2 q^{2\nu} \phi_N^\nu(c) \phi_{N+1}^{\nu+1}(c) + \phi_{N+1}^{\nu+1} \phi_{N-1}^\nu(c) = a_0^2 q^{2\nu} \phi_{N+1}^{\nu+1}(q^{-1}c) \phi_N^\nu(qc). \quad (49)$$

Proof of theorem 3.3. We show that $f_i(c)$ ($i = 0, 1, 2$) defined by equations (39)–(41) satisfy equation (20). Substituting equations (39)–(41) into the numerator of the right-hand side of equation (20), we have

$$\begin{aligned} 1 + q^{-N} f_2(c) + a_0 q^{\nu-N} f_2(c) f_0(c) &= 1 + q^{-N} f_2(c) \left(1 + a_0^2 q^{2\nu} c^2 \frac{\phi_{N+1}^\nu(c) \phi_N^\nu(qc)}{\phi_{N+1}^\nu(qc) \phi_N^\nu(c)} \right) \\ &= 1 + q^{-N} f_2(c) \frac{\phi_{N+1}^\nu(qc) \phi_N^\nu(c) + a_0^2 q^{2\nu} c^2 \phi_{N+1}^\nu(c) \phi_N^\nu(qc)}{\phi_{N+1}^\nu(qc) \phi_N^\nu(c)} \\ &= 1 + q^{-N} f_2(c) q^{-2N} \frac{\phi_{N+1}^{\nu+1}(qc) \phi_N^{\nu-1}(c)}{\phi_{N+1}^\nu(qc) \phi_N^\nu(c)} = 1 - q^{-2N} \frac{\phi_N^{\nu-1}(qc) \phi_{N+1}^{\nu+1}(qc)}{\phi_N^\nu(qc) \phi_{N+1}^\nu(qc)} \\ &= \frac{\phi_N^\nu(qc) \phi_{N+1}^\nu(qc) - q^{-2N} \phi_N^{\nu-1}(qc) \phi_{N+1}^{\nu+1}(qc)}{\phi_N^\nu(qc) \phi_{N+1}^\nu(qc)} \\ &= -a_0^2 c^2 q^{2(\nu-N)} \frac{\phi_{N+1}^\nu(c) \phi_N^\nu(q^2c)}{\phi_N^\nu(qc) \phi_{N+1}^\nu(qc)} \end{aligned}$$

where we used bilinear equations (42) and (47). The denominator is calculated by using equations (43) and (48) as

$$1 + a_0 q^v f_0(c) + q^{N+1} f_0(c) f_1(c) = \frac{q^{-2N} \phi_N^{v-1}(c) \phi_{N+1}^v(q^2 c)}{\phi_{N+1}^v(qc) \phi_N^{v-1}(qc)}. \quad (50)$$

Then, the right-hand side of equation (20) reduces to

$$\begin{aligned} & q^{N+1} f_1(c) \frac{1 + q^{-N} f_2(c) + a_0 q^{v-N} f_2(c) f_0(c)}{1 + a_0 q^v f_0(c) + q^{N+1} f_0(c) f_1(c)} \\ &= q^{N+1} \frac{1}{a_0 q^{v+N-1}} \frac{\phi_{N+1}^v(qc) \phi_N^{v-1}(c)}{\phi_{N+1}^v(c) \phi_N^{v-1}(qc)} \\ & \quad \times a_0^2 c^2 q^{2(v-N)} \frac{\phi_{N+1}^v(c) \phi_N^v(q^2 c)}{\phi_N^v(qc) \phi_{N+1}^v(qc)} q^{2N} \frac{\phi_{N+1}^v(qc) \phi_N^{v-1}(qc)}{\phi_N^{v-1}(c) \phi_{N+1}^v(q^2 c)} \\ &= a_0 c^2 q^{v+2} \frac{\phi_{N+1}^v(qc) \phi_N^v(q^2 c)}{\phi_{N+1}^v(q^2 c) \phi_N^v(qc)} = f_0(qc). \end{aligned}$$

Thus we have shown that equation (20) follows. Equation (21) is checked by using the bilinear equations (43), (47), (45) and (49). Equation (22) follows automatically. \square

Remark. It should be noted that both $1 + a_i f_i$ and $1 + f_i/a_i$ ($i = 0, 1, 2$) admit a multiplicative formula with respect to ϕ . In fact, we have

$$1 + a_0 q^v f_0(c; v, N) = q^{-2N} \frac{\phi_{N+1}^{v+1}(qc) \phi_N^{v-1}(c)}{\phi_{N+1}^v(qc) \phi_N^v(c)} \quad (51)$$

$$1 + a_1 q^{-v+N} f_1(c; v, N) = \frac{a_0^2 q^{2(v-1)} - q^{2N}}{a_0^2 q^{2(v-1)}} \frac{\phi_{N+1}^{v-1}(qc) \phi_N^v(c)}{\phi_{N+1}^v(c) \phi_N^{v-1}(qc)} \quad (52)$$

$$1 + a_2 q^{-N} f_2(c; v, N) = -\frac{\phi_{N-1}^{v-1}(qc) \phi_{N+1}^v(c)}{\phi_N^{v-1}(c) \phi_N^v(qc)} \quad (53)$$

and

$$1 + \frac{f_0(c; v, N)}{a_0 q^v} = \frac{\phi_N^{v-1}(qc) \phi_{N+1}^{v+1}(c)}{\phi_{N+1}^v(qc) \phi_N^v(c)} \quad (54)$$

$$1 + \frac{f_1(c; v, N)}{a_1 q^{-v+N}} = q^{-2N} c^2 (q^{2N} - a_0^2 q^{2(v-1)}) \frac{\phi_{N+1}^{v-1}(c) \phi_N^v(qc)}{\phi_{N+1}^v(c) \phi_N^{v-1}(qc)} \quad (55)$$

$$1 + \frac{f_2(c; v, N)}{a_2 q^{-N}} = \frac{1}{a_0^2 c^2 q^{2(v-1)}} \frac{\phi_{N+1}^v(qc) \phi_{N-1}^{v-1}(c)}{\phi_N^v(qc) \phi_N^{v-1}(c)} \quad (56)$$

which can be verified directly by using equations (39)–(41) and the bilinear equations (42)–(46).

3.2.2. Proof of proposition 3.4. Our basic idea for proving proposition 3.4 is as follows. Bilinear difference equations are derived from the Plücker relations, which are quadratic identities among determinants whose columns are shifted. Therefore, we first construct such ‘difference formulae’ that relate ‘shifted determinants’ and $\phi_N^v(c)$, by using the contiguity relations of $G_v(c)$. We then derive bilinear difference equations with the aid of difference formulae from proper Plücker relations. We take equation (42) as an example to show this procedure explicitly. (For other equations, see the appendix.)

Let us introduce notations,

$$G_v(q^m c) = G_v^m \quad \phi_N^v(q^m c) = \phi_N^{v,m} \tag{57}$$

$$\phi_N^{v,m} = \begin{vmatrix} G_v^m & G_{v+1}^m & \cdots & G_{v+N-1}^m \\ G_{v-1}^m & G_v^m & \cdots & G_{v+N-2}^m \\ \vdots & \vdots & \ddots & \vdots \\ G_{v-N+1}^m & G_{v-N+2}^m & \cdots & G_v^m \end{vmatrix} = |\mathbf{0}^m, \mathbf{1}^m, \dots, \mathbf{N} - \mathbf{1}^m| \tag{58}$$

where \mathbf{k}^m denotes a column vector,

$$\mathbf{k}^m = \begin{pmatrix} G_{v+k}^m \\ G_{v+k-1}^m \\ \vdots \\ G_{v+k-N+1}^m \end{pmatrix}. \tag{59}$$

Here the height of the column is N , but we use the same symbol for determinants with different size, since there is no possibility of confusion.

We next construct a difference formula.

Lemma 3.5 (Difference formula I).

$$|\mathbf{0}^{m+1'}, \mathbf{0}^m, \mathbf{1}^m, \dots, \mathbf{N} - \mathbf{3}^m, \mathbf{N} - \mathbf{2}^m| = (a_0 c q^{m+v-1})^{-2(N-1)} \phi_N^{v,m+1} \tag{60}$$

$$|\mathbf{1}^{m+1'}, \mathbf{0}^m, \mathbf{1}^m, \dots, \mathbf{N} - \mathbf{3}^m, \mathbf{N} - \mathbf{2}^m| = (a_0 c q^{m+v-1})^{-2(N-1)} \phi_N^{v,m+1} \tag{61}$$

where $\mathbf{k}^{m'}$ is a column vector,

$$\mathbf{k}^{m'} = \begin{pmatrix} G_{v+k}^m \\ q^2 G_{v+k-1}^m \\ \vdots \\ q^{2(N-1)} G_{v+k-N+1}^m \end{pmatrix}. \tag{62}$$

Proof of lemma 3.5. We use the contiguity relation (32), which is rewritten as

$$G_{v+1}^{m+1} = G_v^{m+1} + a_0^2 q^{2(v+m)} c^2 G_v^m. \tag{63}$$

in the present notation. Then we have

$$\begin{aligned} \phi_N^{v,m+1} &= \begin{vmatrix} G_v^{m+1} & G_{v+1}^{m+1} & \cdots & G_{v+N-1}^{m+1} \\ G_{v-1}^{m+1} & G_v^{m+1} & \cdots & G_{v+N-2}^{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ G_{v-N+1}^{m+1} & G_{v-N+2}^{m+1} & \cdots & G_v^{m+1} \end{vmatrix} \\ &= \begin{vmatrix} G_v^{m+1} & G_{v+1}^{m+1} - G_v^{m+1} & \cdots & G_{v+N-1}^{m+1} - G_{v+N-2}^{m+1} \\ G_{v-1}^{m+1} & G_v^{m+1} - G_{v-1}^{m+1} & \cdots & G_{v+N-2}^{m+1} - G_{v+N-3}^{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ G_{v-N+1}^{m+1} & G_{v-N+2}^{m+1} - G_{v-N+1}^{m+1} & \cdots & G_v^{m+1} - G_{v-1}^{m+1} \end{vmatrix} \\ &= \begin{vmatrix} G_v^{m+1} & a_0^2 c^2 q^{2(v+m)} G_v^m & \cdots & a_0^2 q^{2(v+N-2+m)} c^2 G_{v+N-2}^m \\ G_{v-1}^{m+1} & a_0^2 c^2 q^{2(v-1+m)} G_{v-1}^m & \cdots & a_0^2 q^{2(v+N-3+m)} c^2 G_{v+N-3}^m \\ \vdots & \vdots & \ddots & \vdots \\ G_{v-N+1}^{m+1} & a_0^2 c^2 q^{2(v-N+1+m)} G_{v-N+1}^m & \cdots & a_0^2 q^{2(v-1+m)} c^2 G_{v-1}^m \end{vmatrix} \\ &= (a_0 c q^{v+m-1})^{2(N-1)} \begin{vmatrix} G_v^{m+1} & G_v^m & \cdots & G_{v+N-2}^m \\ q^2 G_{v-1}^{m+1} & G_{v-1}^m & \cdots & G_{v+N-3}^m \\ \vdots & \vdots & \ddots & \vdots \\ q^{2(N-1)} G_{v-N+1}^{m+1} & G_{v-N+1}^m & \cdots & G_{v-1}^m \end{vmatrix} \\ &= (a_0 c q^{v+m-1})^{2(N-1)} |\mathbf{0}^{m+1'}, \mathbf{0}^m, \mathbf{1}^m, \dots, \mathbf{N} - \mathbf{3}^m, \mathbf{N} - \mathbf{2}^m| \end{aligned}$$

which is equation (60). In the second line of the above calculation, adding the second column to the first column, we obtain

$$\phi_N^{v,m+1} = \begin{vmatrix} G_{v+1}^{m+1} & G_{v+1}^{m+1} - G_v^{m+1} & \cdots & G_{v+N-1}^{m+1} - G_{v+N-2}^{m+1} \\ G_v^{m+1} & G_v^{m+1} - G_{v-1}^{m+1} & \cdots & G_{v-N-2}^{m+1} - G_{v+N-3}^{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ G_{v-N+2}^{m+1} & G_{v-N+2}^{m+1} - G_{v-N+1}^{m+1} & \cdots & G_v^{m+1} - G_{v-1}^{m+1} \end{vmatrix} \tag{64}$$

from which we obtain equation (61) by using equation (63). □

We then consider the Plücker relation,

$$\begin{aligned} 0 = & |\varphi_1, \mathbf{1}^{m+1'}, \mathbf{1}^m, \dots, N - \mathbf{1}^m| \times |\mathbf{0}^m, \mathbf{1}^m, \dots, N - \mathbf{1}^m, N^m| \\ & - |\varphi_1, \mathbf{0}^m, \mathbf{1}^m, \dots, N - \mathbf{1}^m| \times |\mathbf{1}^{m+1'}, \mathbf{1}^m, \dots, N - \mathbf{1}^m, N^m| \\ & + |\mathbf{1}^{m+1'}, \mathbf{0}^m, \mathbf{1}^m, \dots, N - \mathbf{1}^m| \times |\varphi_1, \mathbf{1}^m, \dots, N - \mathbf{1}^m, N^m| \end{aligned} \tag{65}$$

where

$$\varphi_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{66}$$

By expansion with respect to the column φ_1 , the Plücker relation (65) is rewritten as

$$\begin{aligned} 0 = & q^2 |\mathbf{0}^{m+1'}, \mathbf{0}^m, \dots, N - \mathbf{2}^m, N - \mathbf{1}^m| \times |\mathbf{0}^m, \mathbf{1}^m, \dots, N - \mathbf{1}^m| \\ & - |-\mathbf{1}^m, \mathbf{0}^m, \dots, N - \mathbf{2}^m, N - \mathbf{1}^m| \times |\mathbf{1}^{m+1'}, \mathbf{1}^m, \dots, N - \mathbf{1}^m| \\ & + |\mathbf{1}^{m+1'}, \mathbf{0}^m, \mathbf{1}^m, \dots, N - \mathbf{1}^m| \times |\mathbf{0}^m, \dots, N - \mathbf{2}^m, N - \mathbf{1}^m| \end{aligned}$$

where we have used

$$\begin{aligned} |\varphi_1, \mathbf{1}^{m+1'}, \mathbf{1}^m, \dots, N - \mathbf{1}^m| &= \begin{vmatrix} 1 & G_{v+1}^{m+1} & G_{v+1}^m & \cdots & G_{v+N-1}^m \\ 0 & q^2 G_v^{m+1} & G_v^m & \cdots & G_{v+N-2}^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & q^{2N} G_{v-N+1}^{m+1} & G_{v-N+2}^m & \cdots & G_{v-1}^m \end{vmatrix} \\ &= \begin{vmatrix} q^2 G_v^{m+1} & G_v^m & \cdots & G_{v+N-2}^m \\ \vdots & \vdots & \ddots & \vdots \\ q^{2N} G_{v-N+1}^{m+1} & G_{v-N+2}^m & \cdots & G_{v-1}^m \end{vmatrix} \\ &= q^2 |\mathbf{0}^{m+1'}, \mathbf{0}^m, \dots, N - \mathbf{2}^m, N - \mathbf{1}^m|. \end{aligned}$$

Then, we obtain by using lemma 3.5,

$$\begin{aligned} 0 = & q^2 \times (a_0 c q^{v+m-1})^{-2(N-1)} \phi_N^{v,m+1} \times \phi_{N+1}^{v,m} - \phi_N^{v-1,m} \times (a_0 c q^{v+m})^{-2N} \phi_{N+1}^{v+1,m+1} \\ & + (a_0 c q^{v+m-1})^{-2N} \phi_{N+1}^{v,m+1} \times \phi_N^{v,m} \end{aligned} \tag{67}$$

which is the same as equation (42). This completes the proof of the bilinear equation (42).

3.2.3. Determinant formula for negative N . Proposition 3.3 is a determinant formula for the solutions with $N \in \mathbb{Z}_{\geq 0}$, which are obtained by successive application of T_2 on the seed

solution described in proposition 3.1. In order to obtain solutions for $N \in \mathbb{Z}_{<0}$, we have to apply T_2^{-1} on the seed solution; however, we find that this procedure collapses. Therefore, we construct ‘seed and higher solutions’ for negative N so that bilinear equations described in proposition 3.4 hold for all $N \in \mathbb{Z}_{<0}$. To find the seed solutions for $N = -1$, we put $N = -1$, $\phi_{-1}^v(c) = \overline{G}_v(c)$ and $\phi_0^v(c) = 1$ in equations (42)–(49); we obtain a set of contiguity relations to be satisfied by $\overline{G}_v(c)$,

$$a_0^2 c^2 q^{2v} \overline{G}_v(qc) + \overline{G}_v(c) = q^2 \overline{G}_{v-1}(c) \tag{68}$$

$$\overline{G}_v(qc) = \overline{G}_{v+1}(c) + c^2 \overline{G}_{v+1}(qc). \tag{69}$$

Similarly to the case of $N \geq 0$, the following three-term relations in the direction of c or v are derived from equations (68) and (69):

$$\overline{G}_v(c) = \left(\frac{1}{q^2} - c^2 \right) \overline{G}_v(qc) + a_0^2 q^{2v} c^2 \overline{G}_v(q^2c) \tag{70}$$

$$\overline{G}_v(c) = \frac{1}{q^2} (1 + c^2) \overline{G}_{v+1}(c) - \frac{c^2}{q^4} (1 - a_0^2 q^{2v}) \overline{G}_{v+2}(c). \tag{71}$$

Then we obtain a determinant formula for $N < 0$ as follows.

Theorem 3.6. *Let $\overline{G}_v(c)$ be a function satisfying equations (68) and (69). For each $-M = N \in \mathbb{Z}_{<0}$, we define an $M \times M$ Toeplitz determinant*

$$\phi_N^v(c) = \det \left(\overline{G}_{v-i+j}(c) \right)_{i,j=1,\dots,M} \quad (-M = N \in \mathbb{Z}_{<0}). \tag{72}$$

Then, $\phi_N^v(c)$ satisfies the bilinear equations (42)–(49), and therefore equations (39)–(41) satisfy qP_{IV} (20)–(22) with $a_2 = 1$.

Since this theorem is proved by a similar procedure to theorem 3.3, we omit the details.

Remark. If we parametrize as $a_0 = q^\mu$,

$$\overline{G}_v(c; q) = c^2 q^{-2v} G_{-(v-2\mu)}(c; 1/q) \tag{73}$$

satisfies equations (68) and (69) formally.

3.3. q -Okamoto polynomials

In the previous sections we have discussed the particular solutions on the reflection hyperplane of the affine Weyl group $W = W(A_2^{(1)})$ in the parameter space. It is easy to see that qP_{IV} (1) with normalization condition (19) admits a particular solution given by

$$(f_0, f_1, f_2; a_0, a_1, a_2) = (xp^{-1}, xp^{-1}, xp^{-1}; p^{-1}, p^{-1}, p^{-1}) \quad x = c^{2/3} \quad p = q^{-1/3}. \tag{74}$$

Note that this parameter corresponds to the fixed point of π , namely the Dynkin diagram automorphism of the affine Weyl group $W(A_2^{(1)})$. By applying the Bäcklund transformations on this seed solution, we obtain a series of rational solutions on the barycentres of the Weyl chamber of the affine Weyl group $W(A_2^{(1)})$.

Theorem 3.7. *For the parameters*

$$T_1^m T_2^n (a_0, a_1, a_2) = (p^{-3m-1}, p^{3m-3n-1}, p^{3n-1}) \quad m, n \in \mathbb{Z} \tag{75}$$

we have the following rational solutions of the qP_{IV} equation (1):

$$\begin{aligned} T_1^m T_2^n(f_0) &= xp^{m-2n-1} \frac{Q_{m+1,n}(xp^{-2})Q_{m+1,n+1}(x)}{Q_{m+1,n}(x)Q_{m+1,n+1}(xp^{-2})} \\ T_1^m T_2^m(f_1) &= xp^{m+n-1} \frac{Q_{m+1,n+1}(xp^{-2})Q_{m,n}(x)}{Q_{m+1,n+1}(x)Q_{m,n}(xp^{-2})} \\ T_1^m T_2^n(f_2) &= xp^{n-2m-1} \frac{Q_{m,n}(xp^{-2})Q_{m+1,n}(x)}{Q_{m,n}(x)Q_{m+1,n}(xp^{-2})}. \end{aligned} \quad (76)$$

Here $Q_{m,n}(x)$ are monic polynomials defined through the recurrence relations,

$$\begin{aligned} Q_{m-1,n}(x)Q_{m+1,n+1}(xp^{-2}) &= Q_{m,n}(x)Q_{m,n+1}(xp^{-2}) + xp^{-2m-2n}Q_{m,n+1}(x)Q_{m,n}(xp^{-2}) \\ Q_{m+1,n}(x)Q_{m,n+1}(xp^{-2}) &= Q_{m+1,n+1}(x)Q_{m,n}(xp^{-2}) + xp^{4m-2n-2}Q_{m,n}(x)Q_{m+1,n+1}(xp^{-2}) \\ Q_{m+1,n+1}(x)Q_{m,n-1}(xp^{-2}) &= Q_{m,n}(x)Q_{m+1,n}(xp^{-2}) + xp^{4n-2m-2}Q_{m+1,n}(x)Q_{m,n}(xp^{-2}) \end{aligned}$$

with initial conditions $Q_{0,0}(x) = Q_{1,0}(x) = Q_{1,1}(x) = 1$.

We call the polynomials $Q_{m,n}(x)$ the q -Okamoto polynomials. Some examples are as follows.

$$\begin{aligned} Q_{2,0}(x) &= x^2 + (p^2 + 1)x + 1 \\ Q_{3,0}(x) &= x^6 + x^5(p^6 + p^4 + p^2 + 2 + p^{-2}) \\ &\quad + x^4(2p^8 + 2p^6 + 3p^4 + 3p^2 + 3 + p^{-2} + p^{-4}) \\ &\quad + x^3(p^{10} + 2p^8 + 3p^6 + 4p^4 + 4p^2 + 3 + 2p^{-2} + p^{-4}) \\ &\quad + x^2(2p^8 + 2p^6 + 3p^4 + 3p^2 + 3 + p^{-2} + p^{-4}) \\ &\quad + x(p^6 + p^4 + p^2 + 2 + p^{-2}) + 1 \\ Q_{2,1}(x) &= x + 1 \\ Q_{3,1}(x) &= x^4 + x^3(p^4 + p^2 + 1 + p^{-2}) + x^2(2p^4 + p^2 + 2 + p^{-2}) \\ &\quad + x(p^4 + p^2 + 1 + p^{-2}) + 1. \end{aligned}$$

Similarly to the continuous case, q -Okamoto polynomials admit a determinant formula of Jacobi–Trudi type. The proof of the theorem 3.7 as well as the Jacobi–Trudi type formulae will be given in our next paper [9], where we shall discuss the τ functions in a more general setting.

4. Discussion

4.1. $\tilde{W}(A_2^{(1)}) \times \tilde{W}(A_1^{(1)})$ symmetry and comparison with Sakai's $Mul.6$ system

Along with the transformations s_i and π of $\tilde{W}(A_2^{(1)})$, we define the transformations w_0, w_1 and r acting on $\mathbb{C}(a_i, f_i)$ ($i = 0, 1, 2$) as follows:

$$\begin{aligned} w_0(f_0) &= \frac{a_0 a_1 (a_2 a_0 + a_2 f_0 + f_2 f_0)}{f_2 (a_0 a_1 + a_0 f_1 + f_0 f_1)} & w_1(f_0) &= \frac{1 + a_0 f_0 + a_0 a_1 f_0 f_1}{a_0 a_1 f_1 (1 + a_2 f_2 + a_2 a_0 f_2 f_0)} \\ w_0(f_1) &= \frac{a_1 a_2 (a_0 a_1 + a_0 f_1 + f_0 f_1)}{f_0 (a_1 a_2 + a_1 f_2 + f_1 f_2)} & w_1(f_1) &= \frac{1 + a_1 f_1 + a_1 a_2 f_1 f_2}{a_1 a_2 f_2 (1 + a_0 f_0 + a_0 a_1 f_0 f_1)} \\ w_0(f_2) &= \frac{a_2 a_0 (a_1 a_2 + a_1 f_2 + f_1 f_2)}{f_1 (a_2 a_0 + a_2 f_0 + f_2 f_0)} & w_1(f_2) &= \frac{1 + a_2 f_2 + a_2 a_0 f_2 f_0}{a_2 a_0 f_0 (1 + a_1 f_1 + a_1 a_2 f_1 f_2)} \\ r(f_i) &= \frac{1}{f_i} & w_0(a_i) = w_1(a_i) = r(a_i) &= a_i \quad (i = 0, 1, 2). \end{aligned} \quad (77)$$

Lemma 4.1. *The transformations w_0 , w_1 and r generate the extended affine Weyl group $\tilde{W}(A_1^{(1)})$; namely, we have*

$$w_0^2 = w_1^2 = r^2 = 1 \quad rw_0 = w_1r. \tag{78}$$

Moreover, this action of $\tilde{W}(A_1^{(1)}) = \langle w_0, w_1, r \rangle$ commutes with that of $\tilde{W}(A_2^{(1)}) = \langle s_0, s_1, s_2, \pi \rangle$.

Note that the discrete time evolution of the qP_{IV} system is a translation of the $\tilde{W}(A_1^{(1)})$, that is,

$$t = T_4 = rw_0. \tag{79}$$

Lemma 4.2. *The representation of $\tilde{W}(A_2^{(1)}) \times \tilde{W}(A_1^{(1)})$ is equivalent to Sakai's system Mul.6.*

Proof. Let x, y, z and $\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, b_1, b_0 = \tilde{a}_0\tilde{a}_1\tilde{a}_2/b_1$ be Sakai's homogeneous variables and the parameters [17]. Then his representation of $\tilde{W}(A_2^{(1)}) \times \tilde{W}(A_1^{(1)})$ is given as follows:

$$\pi = \sigma^4 \quad s_2 = \pi s_1 \pi^2 \quad s_0 = \pi^2 s_1 \pi \quad r = \sigma^3 \quad w_0 = r w_1 r \tag{80}$$

where σ, s_1 and w_1 are given by⁴

$$\begin{aligned} \sigma(\tilde{a}_0) &= \tilde{a}_1 & \sigma(\tilde{a}_1) &= \tilde{a}_2 & \sigma(\tilde{a}_2) &= \tilde{a}_0 & \sigma(b_0) &= b_1 & \sigma(b_1) &= b_0 \\ \sigma(x) &= \tilde{a}_2 x y (z - x) & \sigma(y) &= -b_1 y z (x + y - z) & \sigma(z) &= \tilde{a}_2 x (x - z)^2 \\ s_1(\tilde{a}_0) &= \tilde{a}_0 \tilde{a}_1 & s_1(\tilde{a}_1) &= \tilde{a}_1^{-1} & s_1(\tilde{a}_2) &= \tilde{a}_2 \tilde{a}_1 & s_1(b_i) &= b_i \quad (i = 0, 1) \\ s_1(x) &= x & s_1(y) &= \tilde{a}_1 y & s_1(z) &= \tilde{a}_1 z \\ w_1(\tilde{a}_i) &= \tilde{a}_i \quad (i = 0, 1, 2) & w_1(b_0) &= b_0 b_1^2 & w_1(b_1) &= b_1^{-1} \\ w_1(x) &= x(y + \tilde{a}_2 x) & w_1(y) &= -b_1 y (y + \tilde{a}_2 x) & w_1(z) &= z(\tilde{a}_2 x - b_1 y). \end{aligned} \tag{81}$$

Introduce variables \tilde{f}_i ($i = 0, 1, 2$) as

$$\tilde{f}_1 = -\frac{x}{z} \quad \tilde{f}_2 = \tilde{a}_2 \frac{x - z}{y} \quad \tilde{f}_0 = \frac{b_1 y z}{\tilde{a}_2 x (x - z)}. \tag{82}$$

Then we have

$$\begin{aligned} \sigma(\tilde{f}_0) &= \frac{\tilde{a}_1}{\tilde{f}_1} & \sigma(\tilde{f}_1) &= \frac{\tilde{a}_2}{\tilde{f}_2} & \sigma(\tilde{f}_2) &= \frac{\tilde{a}_0}{\tilde{f}_0} \\ s_1(\tilde{f}_0) &= \tilde{f}_0 \frac{\tilde{a}_1(1 + \tilde{f}_1)}{\tilde{a}_1 + \tilde{f}_1} & s_1(\tilde{f}_1) &= \frac{\tilde{f}_1}{\tilde{a}_1} & s_1(\tilde{f}_2) &= \frac{\tilde{a}_1 + \tilde{f}_2}{1 + \tilde{f}_1} \\ w_1(\tilde{f}_0) &= \frac{1 + \tilde{f}_0 + \tilde{f}_0 \tilde{f}_1}{\tilde{f}_1(1 + \tilde{f}_2 + \tilde{f}_2 \tilde{f}_0)} & w_1(\tilde{f}_1) &= \frac{1 + \tilde{f}_1 + \tilde{f}_1 \tilde{f}_2}{\tilde{f}_2(1 + \tilde{f}_0 + \tilde{f}_0 \tilde{f}_1)} \\ w_1(\tilde{f}_2) &= \frac{1 + \tilde{f}_2 + \tilde{f}_2 \tilde{f}_0}{\tilde{f}_0(1 + \tilde{f}_1 + \tilde{f}_1 \tilde{f}_2)}. \end{aligned} \tag{83}$$

This representation is equivalent to our representation by the relation $\tilde{f}_i = a_i f_i$, $\tilde{a}_i = a_i^2$. □

⁴ The generators σ, s_1, w_1 here correspond to $\sigma_{(123450)}$, w_1 and w'_1 in [17]. An error in the formula for $\sigma_{(123450)}$ [17] is corrected.

4.2. Ultra-discretization of qP_{IV}

In section 2.3, we mentioned that qP_{IV} admit a continuous limit to the symmetric form of P_{IV} . There is another interesting limit, which is known as the ‘ultra-discrete limit’ [18]. We put

$$f_i = e^{F_i/\varepsilon} \quad a_i = e^{A_i/\varepsilon} \quad i = 0, 1, 2$$

and take the limit $\varepsilon \rightarrow +0$. By using the formula

$$\lim_{\varepsilon \rightarrow +0} \varepsilon \log \left(e^{\frac{A}{\varepsilon}} + e^{\frac{B}{\varepsilon}} + \dots \right) = \max(A, B, \dots) \tag{84}$$

qP_{IV} (1) yields

$$\begin{aligned} \bar{F}_0 &= t(F_0) = A_0 + A_1 + F_1 + \max(0, A_2 + F_2, A_0 + A_2 + F_0 + F_2) \\ &\quad - \max(0, A_0 + F_0, A_1 + A_0 + F_1 + F_0) \\ \bar{F}_1 &= t(F_1) = A_1 + A_2 + F_2 + \max(0, A_0 + F_0, A_1 + A_0 + F_1 + F_0) \\ &\quad - \max(0, A_1 + F_1, A_2 + A_1 + F_2 + F_1) \\ \bar{F}_2 &= t(F_2) = A_2 + A_0 + F_0 + \max(0, A_1 + F_1, A_2 + A_1 + F_2 + F_1) \\ &\quad - \max(0, A_2 + F_2, A_0 + A_2 + F_0 + F_2) \\ \bar{A}_i &= t(A_i) = A_i \end{aligned} \tag{85}$$

which we call the fourth ultra-discrete Painlevé equation(uP_{IV}). Simultaneously, Bäcklund transformation (6) is ultra-discretized as

$$\begin{aligned} s_i(F_j) &= F_j + u_{ij} (\max(A_i, F_i) - \max(0, A_i + F_i)) \\ s_i(A_j) &= A_j - a_{ij} A_i \quad \pi(X_i) = X_{i+1} \quad (i = 0, 1, 2) \quad X = F, A \end{aligned} \tag{86}$$

where $A = (a_{ij})_{i,j=0,1,2}$ and $U = (u_{ij})_{i,j=0,1,2}$ are given by equation (7). Then we can verify the following:

Proposition 4.3. *The transformations s_0, s_1, s_2 and π of the A -variables and the F -variables, defined by equation (86), generate the extended affine Weyl group $\tilde{W} = \langle s_0, s_1, s_2, \pi \rangle$ of type $A_2^{(1)}$. Furthermore, they commute with the time evolution t of the fourth ultra-discrete Painlevé equation uP_{IV} .*

Appendix. Difference formulae and Plücker relations

In this appendix, we provide data which are necessary for the proof of proposition 3.4.

We first note that it is possible to express $\phi_N^{v,m}$ in section 3.2 as a Casorati determinant with respect to m as follows.

Lemma A.1. $\phi_N^{v,m}$ is rewritten as

$$\phi_N^{v,m} = \begin{vmatrix} G_v^m & G_{v-1}^m & \dots & G_{v-N+1}^m \\ G_v^{m+1} & G_{v-1}^{m+1} & \dots & G_{v-N+1}^{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ G_v^{m+N-1} & G_{v-1}^{m+N-1} & \dots & G_{v-N+1}^{m+N-1} \end{vmatrix} \tag{A.1}$$

$$\phi_N^{v,m} = \prod_{k=1}^{N-1} \left(\frac{a_0^2 q^{2(n-k)}}{a_0^2 q^{2(n-k)} - 1} \right)^{N-k} \begin{vmatrix} G_v^m & G_v^{m+1} & \dots & G_v^{m+N-1} \\ G_v^{m-1} & G_v^m & \dots & G_v^{m+N-2} \\ \vdots & \vdots & \ddots & \vdots \\ G_v^{m-N+1} & G_v^{m-N+2} & \dots & G_v^m \end{vmatrix}. \tag{A.2}$$

Proof of lemma A.1. We use the contiguity relation (33), which is rewritten with the notation introduced in equation (57) as

$$G_v^{m+1} = G_{v+1}^m - q^{2m} c^2 G_v^m. \tag{A.3}$$

For $k = 2$ to N , subtracting the $(j - 1)$ th column multiplied by $q^{2m} c^2$ from the j th column of equation (58) for $j = N, \dots, k$, we obtain

$$\begin{aligned} \phi_N^{v,m} &= \begin{vmatrix} G_v^m & G_{v+1}^m - q^{2m} c^2 G_v^m & \cdots & G_{v+N-1}^m - q^{2m} c^2 G_{v+N-2}^m \\ G_{v-1}^m & G_v^m - q^{2m} c^2 G_{v-1}^m & \cdots & G_{v+N-2}^m - q^{2m} c^2 G_{v+N-3}^m \\ \vdots & \vdots & \ddots & \vdots \\ G_{v-N+1}^m & G_{v-N+2}^m - q^{2m} c^2 G_{v-N+1}^m & \cdots & G_v^m - q^{2m} c^2 G_{v-1}^m \end{vmatrix} \\ &= \begin{vmatrix} G_v^m & G_v^{m+1} & \cdots & G_{v+N-2}^{m+1} \\ G_{v-1}^m & G_{v-1}^{m+1} & \cdots & G_{v+N-3}^{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ G_{v-N+1}^m & G_{v-N+1}^{m+1} & \cdots & G_{v-1}^{m+1} \end{vmatrix} \\ &= \cdots \\ &= \begin{vmatrix} G_v^m & G_v^{m+1} & \cdots & G_v^{m+N-1} \\ G_{v-1}^m & G_{v-1}^{m+1} & \cdots & G_{v-1}^{m+N-1} \\ \vdots & \vdots & \ddots & \vdots \\ G_{v-N+1}^m & G_{v-N+1}^{m+1} & \cdots & G_{v-N+1}^{m+N-1} \end{vmatrix} \end{aligned} \tag{A.4}$$

which yields equation (A.1) by taking the transposition. In order to derive equation (A.2), we use the contiguity relation,

$$G_{v-1}^m + \frac{G_v^m}{a_0^2 q^{2(v-1)} - 1} = \frac{a_0^2 q^{2(v-1)}}{a_0^2 q^{2(v-1)} - 1} G_v^{m-1} \tag{A.5}$$

which is obtained from equations (32) and (33). For $k = 2$ to N , subtracting the $(i - 1)$ th row multiplied by $\frac{1}{a_0^2 q^{2(v-i-1)} - 1}$ from the i th row of equation (A.4) for $i = N, \dots, k$, we obtain

$$\begin{aligned} \phi_N^{v,m} &= \begin{vmatrix} G_v^m & G_v^{m+1} & \cdots & G_v^{m+N-1} \\ \vdots & \vdots & \ddots & \vdots \\ G_{v-N+2}^m & G_{v-N+2}^{m+1} & \cdots & G_{v-N+2}^{m+N-1} \\ G_{v-N+1}^m - \frac{G_{v-N+2}^m}{a_0^2 q^{2(v-N+1)} - 1} & G_{v-N+1}^{m+1} - \frac{G_{v-N+2}^{m+1}}{a_0^2 q^{2(v-N+1)} - 1} & \cdots & G_{v-N+1}^{m+N-1} - \frac{G_{v-N+2}^{m+N-1}}{a_0^2 q^{2(v-N+1)} - 1} \end{vmatrix} \\ &= \frac{a_0^2 q^{2(v-N+1)}}{a_0^2 q^{2(v-N+1)} - 1} \begin{vmatrix} G_v^m & G_v^{m+1} & \cdots & G_v^{m+N-1} \\ \vdots & \vdots & \ddots & \vdots \\ G_{v-N+2}^m & G_{v-N+2}^{m+1} & \cdots & G_{v-N+2}^{m+N-1} \\ G_{v-N+2}^{m-1} & G_{v-N+2}^m & \cdots & G_{v-N+2}^{m+N-2} \end{vmatrix} \\ &= \cdots \\ &= \prod_{i=1}^{N-1} \frac{a_0^2 q^{2(v-i)}}{a_0^2 q^{2(v-i)} - 1} \begin{vmatrix} G_v^m & G_v^{m+1} & \cdots & G_v^{m+N-1} \\ G_{v-1}^{m-1} & G_{v-1}^m & \cdots & G_{v-1}^{m+N-2} \\ \vdots & \vdots & \ddots & \vdots \\ G_{v-N+2}^{m-1} & G_{v-N+2}^m & \cdots & G_{v-N+2}^{m+N-2} \end{vmatrix} \\ &= \cdots \end{aligned}$$

$$= \prod_{i=1}^{N-1} \left(\frac{a_0^2 q^{2(v-i)}}{a_0^2 q^{2(v-i)} - 1} \right)^{N-i} \begin{vmatrix} G_v^m & G_v^{m+1} & \cdots & G_v^{m+N-1} \\ G_v^{m-1} & G_v^m & \cdots & G_v^{m+N-2} \\ \vdots & \vdots & \ddots & \vdots \\ G_v^{m-N+2} & G_v^{m-N+3} & \cdots & G_v^{m-1} \\ G_v^{m-N+1} & G_v^{m-N} & \cdots & G_v^m \end{vmatrix}$$

which is equation (A.2). □

In view of this lemma, we introduce the following notations.

$$\phi_N^{v,m} = \begin{vmatrix} G_v^m & G_{v+1}^m & \cdots & G_{v+N-1}^m \\ G_{v-1}^m & G_v^m & \cdots & G_{v-N-2}^m \\ \vdots & \vdots & \ddots & \vdots \\ G_{v-N+1}^m & G_{v-N+2}^m & \cdots & G_v^m \end{vmatrix} = |\mathbf{0}^m, \mathbf{1}^m, \dots, N - \mathbf{1}^m| \tag{A.6}$$

$$\phi_N^{v,m} = \begin{vmatrix} G_v^m & G_{v-1}^m & \cdots & G_{v-N+1}^m \\ G_v^{m+1} & G_{v-1}^{m+1} & \cdots & G_{v-N+1}^{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ G_v^{m+N-1} & G_{v-1}^{m+N-1} & \cdots & G_{v-N+1}^{m+N-1} \end{vmatrix} = |\underline{\mathbf{0}}^m, \underline{-\mathbf{1}}^m, \dots, \underline{-N + \mathbf{1}}^m| \tag{A.7}$$

$$\psi_N^{v,m} = \begin{vmatrix} G_v^m & G_v^{m+1} & \cdots & G_v^{m+N-1} \\ G_v^{m-1} & G_v^m & \cdots & G_v^{m+N-2} \\ \vdots & \vdots & \ddots & \vdots \\ G_v^{m-N+1} & G_v^{m-N+2} & \cdots & G_v^m \end{vmatrix} = |\overline{\mathbf{0}}_v, \overline{\mathbf{1}}_v, \dots, \overline{N - \mathbf{2}}_v, \overline{N - \mathbf{1}}_v| \tag{A.8}$$

where \mathbf{k}^m , $\underline{\mathbf{k}}^m$ and $\overline{\mathbf{k}}_v$ are column vectors given by

$$\mathbf{k}^m = \begin{pmatrix} G_{v+k}^m \\ G_{v+k-1}^m \\ \vdots \\ G_{v+k-N+1}^m \end{pmatrix} \quad \underline{\mathbf{k}}^m = \begin{pmatrix} G_{v+k}^m \\ G_{v+k}^{m+1} \\ \vdots \\ G_{v+k}^{m+N-1} \end{pmatrix} \quad \overline{\mathbf{k}}_v = \begin{pmatrix} G_v^{m+k} \\ G_v^{m+k-1} \\ \vdots \\ G_v^{m+k-N+1} \end{pmatrix} \tag{A.9}$$

respectively. We also use auxiliary column vectors,

$$\mathbf{k}^{m'} = \begin{pmatrix} G_{v+k}^m \\ q^2 G_{v+k-1}^m \\ \vdots \\ q^{2(N-1)} G_{v+k-N+1}^m \end{pmatrix} \quad \underline{\mathbf{k}}^{m\ddagger} = \begin{pmatrix} G_{v+k}^m \\ q^{-2} G_{v+k}^{m+1} \\ \vdots \\ q^{-2(N-1)} G_{v+k}^{m+N-1} \end{pmatrix} \tag{A.10}$$

$$\overline{\mathbf{k}}_v' = \begin{pmatrix} G_v^{m+k} \\ q^2 G_v^{m+k-1} \\ \vdots \\ q^{2(N-1)} G_v^{m+k-N+1} \end{pmatrix}.$$

Now we give difference formulae.

Difference formula I

$$\begin{aligned} |\mathbf{0}^{m+1'}, \mathbf{0}^m, \mathbf{1}^m, \dots, N - \mathbf{3}^m, N - \mathbf{2}^m| &= (a_0 c q^{m+v-1})^{-2(N-1)} \phi_N^{v,m+1} \\ |\mathbf{1}^{m+1'}, \mathbf{0}^m, \mathbf{1}^m, \dots, N - \mathbf{3}^m, N - \mathbf{2}^m| &= (a_0 c q^{m+v-1})^{-2(N-1)} \phi_N^{v,m+1}. \end{aligned} \tag{A.11}$$

Table A.1. Data for the proof of difference formulae.

Difference formula	Contiguity relation	Determinant
Difference formula I (A.11)	$G_{v+1}^{m+1} = G_v^{m+1} + a_0^2 q^{2(v+m)} c^2 G_v^m$	Equation (A.6)
Difference formula II (A.12)	$G_v^{m+1} = G_v^m - q^{2m} c^2 G_v^m$	Equation (A.6)
Difference formula III (A.13)	$G_v^m - G_{v-1}^m = a_0^2 c^2 q^{2(v+m-2)} G_{v-1}^{m-1}$	Equation (A.7)
Difference formula IV (A.14)	$G_{v+1}^{m+1} - G_{v+1}^m = c^2 q^{2m} (1 - a_0 q^{2v}) G_v^m$	Equation (A.8)
Difference formula V (A.15)	$G_{v+1}^{m+1} - a_0 q^{2v} G_{v+1}^m = (1 - a_0 q^{2v}) G_v^{m+1}$	Equation (A.8)

Difference formula II

$$\begin{aligned}
 |\mathbf{0}^m, \mathbf{0}^{m+1}, \mathbf{1}^{m+1}, \dots, N - \mathbf{3}^{m+1}, N - \mathbf{2}^{m+1}| &= \phi_N^{v,m} \\
 |\mathbf{1}^m, \mathbf{0}^{m+1}, \mathbf{1}^{m+1}, \dots, N - \mathbf{3}^{m+1}, N - \mathbf{2}^{m+1}| &= c^2 q^{2m} \phi_N^{v,m}.
 \end{aligned}
 \tag{A.12}$$

Difference formula III

$$\begin{aligned}
 |-\mathbf{1}^{m-1}, -\mathbf{2}^{m-1}, \dots, -N + \mathbf{1}^{m-1}, -N + \mathbf{1}^{m\uparrow}| &= (a_0 c q^{v+m-1})^{-2(N-1)} \phi_N^{v,m} \\
 |-\mathbf{1}^{m-1}, -\mathbf{2}^{m-1}, \dots, -N + \mathbf{1}^{m-1}, -N + \mathbf{2}^{m\uparrow}| &= (a_0 c q^{v+m-1})^{-2(N-1)} \phi_N^{v,m}.
 \end{aligned}
 \tag{A.13}$$

Difference formula IV

$$\begin{aligned}
 |\overline{\mathbf{0}}_{v+1}', \overline{\mathbf{0}}_v, \dots, \overline{N - \mathbf{2}}_v| &= (c q^{m-1})^{-2(N-1)} (a_0^2 q^{2v} - 1)^{-(N-1)} \psi_N^{v+1,m} \\
 |\overline{\mathbf{1}}_{v+1}', \overline{\mathbf{0}}_v, \dots, \overline{N - \mathbf{2}}_v| &= (q^{m-1} c)^{-2(N-1)} (a_0^2 q^{2v} - 1)^{-(N-1)} \psi_N^{v+1,m}.
 \end{aligned}
 \tag{A.14}$$

Difference formula V

$$\begin{aligned}
 |\overline{\mathbf{0}}_{v+1}, \overline{\mathbf{0}}_v, \dots, \overline{N - \mathbf{2}}_v| &= (1 - a_0^2 q^{2v})^{-(N-1)} \psi_N^{v+1,m} \\
 |\overline{\mathbf{1}}_{v+1}, \overline{\mathbf{0}}_v, \dots, \overline{N - \mathbf{2}}_v| &= (1 - a_0^2 q^{2v})^{-(N-1)} a_0^2 q^{2v} \psi_N^{v+1,m}.
 \end{aligned}
 \tag{A.15}$$

Difference formula I has been proved in lemma 3.5. Other formulae are proved in a similar manner. Table A.1 shows the contiguity relation and determinant to be used for the derivation of each difference formula.

Next we give the list of Plücker relations which are necessary for the proof of proposition 3.4.

Plücker relation I

$$\begin{aligned}
 0 = & |\varphi_1, \mathbf{1}^{m+1'}, \mathbf{1}^m, \dots, N - \mathbf{1}^m| \times |\mathbf{0}^m, \mathbf{1}^m, \dots, N - \mathbf{1}^m, N^m| \\
 & - |\varphi_1, \mathbf{0}^m, \mathbf{1}^m, \dots, N - \mathbf{1}^m| \times |\mathbf{1}^{m+1'}, \mathbf{1}^m, \dots, N - \mathbf{1}^m, N^m| \\
 & + |\mathbf{1}^{m+1'}, \mathbf{0}^m, \mathbf{1}^m, \dots, N - \mathbf{1}^m| \times |\varphi_1, \mathbf{1}^m, \dots, N - \mathbf{1}^m, N^m|.
 \end{aligned}
 \tag{A.16}$$

Plücker relation II

$$\begin{aligned}
 0 = & |\varphi_2, \mathbf{1}^{m+1'}, \mathbf{1}^m, \dots, N - \mathbf{1}^m| \times |\mathbf{0}^m, \mathbf{1}^m, \dots, N - \mathbf{1}^m, N^m| \\
 & - |\varphi_2, \mathbf{0}^m, \mathbf{1}^m, \dots, N - \mathbf{1}^m| \times |\mathbf{1}^{m+1'}, \mathbf{1}^m, \dots, N - \mathbf{1}^m, N^m| \\
 & + |\mathbf{1}^{m+1'}, \mathbf{0}^m, \mathbf{1}^m, \dots, N - \mathbf{1}^m| \times |\varphi_2, \mathbf{1}^m, \dots, N - \mathbf{1}^m, N^m|.
 \end{aligned}
 \tag{A.17}$$

Plücker relation III

$$\begin{aligned}
 0 = & |\varphi_2, \mathbf{1}^m, \mathbf{1}^{m+1}, \dots, N - \mathbf{1}^{m+1}| \times |\mathbf{0}^{m+1}, \mathbf{1}^{m+1}, \dots, N - \mathbf{1}^{m+1}, N^{m+1}| \\
 & - |\varphi_2, \mathbf{0}^{m+1}, \mathbf{1}^{m+1}, \dots, N - \mathbf{1}^{m+1}| \\
 & \times |\mathbf{1}^m, \mathbf{1}^{m+1}, \dots, N - \mathbf{1}^{m+1}, N^{m+1}| \\
 & + |\mathbf{1}^m, \mathbf{0}^{m+1}, \mathbf{1}^{m+1}, \dots, N - \mathbf{1}^{m+1}| \\
 & \times |\varphi_2, \mathbf{1}^{m+1}, \dots, N - \mathbf{1}^{m+1}, N^{m+1}|.
 \end{aligned}
 \tag{A.18}$$

Table A.2. Data for the proof of bilinear equations.

Bilinear equation	Plücker relation	Difference formula
Equation (42)	Plücker relation I (A.16)	Difference formula I (A.11)
Equations (43) and (44)	Plücker relation II (A.17) Plücker relation III (A.18)	Difference formula I (A.11) Difference formula II (A.12)
Equation (45)	Plücker relation IV (A.19)	Difference formula II (A.12)
Equation (46)	Plücker relation V (A.20)	Difference formula I (A.11)
Equation (47)	Plücker relation VI(A.21)	Difference formula III (A.13)
Equation (48)	Plücker relation VII (A.22)	Difference formula IV (A.14)
Equation (49)	Plücker relation VIII(A.23)	Difference formula V (A.15)

Plücker relation IV

$$\begin{aligned}
0 = & |\varphi_2, \varphi_1, \mathbf{1}^{m+1}, \dots, N - \mathbf{1}^{m+1}| \times |\mathbf{1}^m, \mathbf{1}^{m+1}, \dots, N - \mathbf{1}^{m+1}, N^{m+1}| \\
& - |\varphi_2, \mathbf{1}^m, \mathbf{1}^{m+1}, \dots, N - \mathbf{1}^{m+1}| \times |\varphi_1, \mathbf{1}^{m+1}, \dots, N - \mathbf{1}^{m+1}, N^{m+1}| \\
& + |\varphi_1, \mathbf{1}^m, \mathbf{1}^{m+1}, \dots, N - \mathbf{1}^{m+1}| \\
& \times |\varphi_2, \mathbf{1}^{m+1}, \dots, N - \mathbf{1}^{m+1}, N^{m+1}|. \tag{A.19}
\end{aligned}$$

Plücker relation V

$$\begin{aligned}
0 = & |\varphi_2, \varphi_1, \mathbf{0}^m, \dots, N - \mathbf{2}^m| \times |\mathbf{0}^{m+1'}, \mathbf{0}^m, \dots, N - \mathbf{2}^m, N - \mathbf{1}^m| \\
& - |\varphi_2, \mathbf{0}^{m+1'}, \mathbf{0}^m, \dots, N - \mathbf{2}^m| \times |\varphi_1, \mathbf{0}^m, \dots, N - \mathbf{2}^m, N - \mathbf{1}^m| \\
& + |\varphi_1, \mathbf{0}^{m+1'}, \mathbf{0}^m, \dots, N - \mathbf{2}^m| \times |\varphi_2, \mathbf{0}^m, \dots, N - \mathbf{2}^m, N - \mathbf{1}^m|. \tag{A.20}
\end{aligned}$$

Plücker relation VI

$$\begin{aligned}
0 = & |\underline{\mathbf{0}}^{m-1}, \underline{-\mathbf{1}}^{m-1}, \dots, \underline{-N + \mathbf{1}}^{m-1}, \underline{-N}^{m-1}| \\
& \times |\underline{-\mathbf{1}}^{m-1}, \dots, \underline{-N + \mathbf{1}}^{m-1}, \underline{-N + \mathbf{1}}^{m+1\ddagger}, \varphi_1| \\
& - |\underline{\mathbf{0}}^{m-1}, \underline{-\mathbf{1}}^{m-1}, \dots, \underline{-N + \mathbf{1}}^{m-1}, \underline{-N + \mathbf{1}}^{m\ddagger}| \\
& \times |\underline{-\mathbf{1}}^{m-1}, \dots, \underline{-N + \mathbf{1}}^{m-1}, \underline{-N}^{m-1}, \varphi_1| \\
& + |\underline{\mathbf{0}}^{m-1}, \underline{-\mathbf{1}}^{m-1}, \dots, \underline{-N + \mathbf{1}}^{m-1}, \varphi_1| \\
& \times |\underline{-\mathbf{1}}^{m-1}, \dots, \underline{-N + \mathbf{1}}^{m-1}, \underline{-N}^{m-1}, \underline{-N + \mathbf{1}}^{m\ddagger}|. \tag{A.21}
\end{aligned}$$

Plücker relation VII

$$\begin{aligned}
0 = & |\varphi_1, \bar{\mathbf{1}}'_{v+1}, \bar{\mathbf{1}}_v, \dots, \overline{N - \mathbf{1}}_v| \times |\bar{\mathbf{0}}_v, \bar{\mathbf{1}}_v, \dots, \overline{N - \mathbf{1}}_v, \overline{N}_v| \\
& - |\varphi_1, \bar{\mathbf{0}}_v, \bar{\mathbf{1}}_v, \dots, \overline{N - \mathbf{1}}_v| \times |\bar{\mathbf{1}}'_{v+1}, \bar{\mathbf{1}}_v, \dots, \overline{N - \mathbf{1}}_v, \overline{N}_v| \\
& + |\bar{\mathbf{1}}'_{v+1}, \bar{\mathbf{0}}_v, \bar{\mathbf{1}}_v, \dots, \overline{N - \mathbf{1}}_v| \times |\varphi_1, \bar{\mathbf{1}}_v, \dots, \overline{N - \mathbf{1}}_v, \overline{N}_v|. \tag{A.22}
\end{aligned}$$

Plücker relation VIII

$$\begin{aligned}
0 = & |\varphi_2, \varphi_1, \bar{\mathbf{1}}_v, \dots, \overline{N - \mathbf{1}}_v| \times |\bar{\mathbf{0}}_{v+1}, \bar{\mathbf{1}}_v, \dots, \overline{N - \mathbf{1}}_v, \overline{N}_v| \\
& - |\varphi_2, \bar{\mathbf{0}}_{v+1}, \bar{\mathbf{1}}_v, \dots, \overline{N - \mathbf{1}}_v| \times |\varphi_1, \bar{\mathbf{1}}_v, \dots, \overline{N - \mathbf{1}}_v, \overline{N}_v| \\
& + |\varphi_1, \bar{\mathbf{0}}_{v+1}, \bar{\mathbf{1}}_v, \dots, \overline{N - \mathbf{1}}_v| \times |\varphi_2, \bar{\mathbf{1}}_v, \dots, \overline{N - \mathbf{1}}_v, \overline{N}_v|. \tag{A.23}
\end{aligned}$$

Here, φ_i ($i = 1, 2$) are any column vectors which we specialize as

$$\varphi_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad \varphi_2 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{A.24}$$

in the proof of bilinear equations.

In section 3.2.2, we showed how equation (42) can be derived from the Plücker relation I and difference formula I. The other bilinear equations in proposition 3.4 are obtained similarly by using the data described in table A.2.

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